

# A note on the existence of traveling-wave solutions to a Boussinesq system

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**Abstract.** We obtain a one-parameter family

$$(u_\mu(x, t), \eta_\mu(x, t))_{\mu \geq \mu_0} = (\phi_\mu(x - \omega_\mu t), \psi_\mu(x - \omega_\mu t))_{\mu \geq \mu_0}$$

of traveling-wave solutions to the Boussinesq system

$$\begin{cases} u_t + \eta_x + uu_x + c\eta_{xxx} = 0 \\ \eta_t + u_x + (\eta u)_x + au_{xxx} = 0 \end{cases} \quad (x, t) \in \mathbb{R}^2$$

in the case  $a, c < 0$ , with non-null speeds  $\omega_\mu$  arbitrarily close to 0 ( $\omega_\mu \xrightarrow{\mu \rightarrow +\infty} 0$ ). We show that the  $L^2$ -size of such traveling-waves satisfies the uniform (in  $\mu$ ) estimate  $\|\phi_\mu\|_2^2 + \|\psi_\mu\|_2^2 \leq C\sqrt{|a| + |c|}$ , where  $C$  is a positive constant. Furthermore,  $\phi_\mu$  and  $-\psi_\mu$  are smooth, non-negative, radially decreasing functions which decay exponentially at infinity.

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## 1. Introduction

In [1], with the purpose of describing the dynamics of small-amplitude long waves propagating on the surface of an ideal fluid in a channel of constant depth, the authors introduced the four-parameter family of Boussinesq systems

$$\begin{cases} u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0 \\ \eta_t + u_x + (\eta u)_x + au_{xxx} - b_{xxt} = 0. \end{cases} \quad (1.1)$$

These systems are first-order approximations to the Euler equations in the small parameters  $\alpha = \frac{A}{h} \ll 1$  and  $\beta = \frac{h^2}{l^2} \ll 1$ , where  $h$  is the depth of the channel (subsequently scaled to 1), and  $A$  and  $\lambda$  represent a typical wave amplitude and a typical wavelength respectively. Here,  $\eta(x, t)$  denotes the

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deviation of free surface with respect to the undisturbed state (i.e.  $1 + \eta(x, t)$  is the total depth of the liquid at time  $t$  and position  $x$ ) and  $u(x, t)$  is the horizontal velocity at height  $\theta$ ,  $0 \leq \theta \leq 1$ . The four parameters  $a, b, c$  and  $d$  are given by

$$\begin{aligned} a &= \frac{1}{2} \left( \theta^2 - \frac{1}{3} \right) \lambda, & b &= \frac{1}{2} \left( \theta^2 - \frac{1}{3} \right) (1 - \lambda) \\ c &= \frac{1}{2} (1 - \theta^2) \mu, & d &= \frac{1}{2} (1 - \theta^2) (1 - \mu), \end{aligned} \quad (1.2)$$

where, as stated in [1],  $\lambda$  and  $\mu$  are modeling parameters that do not possess a direct physical interpretation.

In [5], a correction of the  $c$  parameter is proposed in order to include the contribution of the surface tension:

$$c = \frac{1}{2} (1 - \theta^2) \mu - \tau. \quad (1.3)$$

The Bond number  $\tau$  is given by  $\tau = \frac{\Gamma}{\rho g h^2}$ , where  $\Gamma$  is the surface tension coefficient and  $\rho$  the density of water.

In [3], the authors prove the existence (and orbital stability) of traveling-waves to system (1.1) of the form

$$u(x, t) = \phi(x - \omega t), \eta(x, t) = \psi(x - \omega t) \quad (1.4)$$

in the case  $a, c < 0$ ,  $b = d > 0$  and  $ac > b^2$ . Furthermore, in [4], the existence of such traveling-waves with small propagation speed is obtained in the case  $a, c < 0$  and  $b = d$ .

In the present paper we exhibit a new family of one-parameter traveling-waves in the case  $a, c < 0$  and  $b = d = 0$ . Our method has the advantage of providing radially decreasing functions and a uniform bound for the  $L^2$ -size of the solution. More precisely, we prove the following result:

*Theorem 1.1.* Let  $a, c < 0$ . There exists a constant  $\mu_0 = \mu_0(a, c)$  and a one-parameter family of nontrivial traveling-wave solutions to the Boussinesq system

$$\begin{cases} u_t + \eta_x + uu_x + c\eta_{xxx} = 0 \\ \eta_t + u_x + (\eta u)_x + au_{xxx} = 0 \end{cases} \quad (1.5)$$

of the form

$$(u, \eta) = (\phi_\mu(x - \omega_\mu t), \psi_\mu(x - \omega_\mu t)), \quad \mu \geq \mu_0,$$

with  $(\phi_\mu, \psi_\mu) \in H^\infty(\mathbb{R}) \times H^\infty(\mathbb{R})$ ,  $\phi_\mu$  and  $-\psi_\mu$  non-negative and radially decreasing, with exponential decay at infinity.

Furthermore, the speed  $\omega_\mu$  satisfies the estimate

$$0 > \omega_\mu > -\frac{1}{C_1} \sqrt[3]{|a| + |c|} \mu^{-\frac{2}{3}}$$

and the following uniform control of the  $L^2$ -norm of the traveling-wave holds:

$$\|\phi_\mu\|_2^2 + \|\psi_\mu\|_2^2 \leq C \sqrt{|a| + |c|}, \quad (1.6)$$

where  $C$  is a positive constant independent of  $a$ ,  $c$  and  $\mu$ .

*Remark 1.2.* We stated the existence of traveling-waves

$$(\phi(x - \omega t), \psi(x - \omega t)),$$

with  $\phi \geq 0$  and  $\psi \leq 0$ , propagating with negative speed  $\omega$ . Noticing that  $(-\phi(x + \omega t), \psi(x + \omega t))$  is also a solution to (1.5), it is straightforward to deduce the existence of non-positive traveling-waves propagating with positive speed.

*Remark 1.3.* As stated above, in [4], the authors also establish the existence of traveling-waves with small speed in the particular case  $a, c < 0$  and  $b = d = 0$ , although it is not clear, with the method used, if the solutions have a sign, are radially-decreasing or if an uniform (in the speed  $\omega$ ) estimate such as (1.6) holds. On the other hand, of course, the method used in [4] has the important advantage of covering the case  $b = d \neq 0$ . Either way, it does not seem obvious to prove or to disprove that the traveling-waves found in both papers are the same.

*Remark 1.4.* Note, in view of (1.2) and (1.3), that the case treated in Theorem 1.1 corresponds to  $\lambda = \mu = 1$ , that is

$$a = \frac{1}{2} \left( \theta^2 - \frac{1}{3} \right) \quad \text{and} \quad c = \frac{1}{2} (1 - \theta^2) - \tau.$$

For  $\theta^2 \rightarrow \frac{1}{3}$  and  $\tau \rightarrow \frac{1}{3}$  we get  $a, c \rightarrow 0$ . The estimate (1.6) then suggests that the traveling-wave solutions vanish in this regime.

This is consistent with the known fact (see [5]) that in the case  $\frac{1}{3} - \tau = \mathcal{O}(\beta)$ ,  $\beta \rightarrow 0$  it is necessary to introduce higher order terms in the Boussinesq approximation of the Euler equations in order to model solitary waves.

## 2. The minimization problem

In what follows we fix  $a, c < 0$ . In order to prove Theorem 1.1 we introduce a variational problem whose minimizers, up to a rescaling, correspond to a traveling-wave profile. Let us first note that

$$(u, \eta) = (\phi(x - \omega t), \psi(x - \omega t)), \quad \phi(y), \psi(y) \xrightarrow{y \rightarrow \infty} 0$$

is a solution to the Boussinesq system (1.5) if and only if  $\phi$  and  $\psi$  satisfy the stationary equation

$$\begin{cases} a\phi'' + \phi - \omega\psi + \phi\psi = 0 \\ c\psi'' + \psi - \omega\phi + \frac{1}{2}\phi^2 = 0. \end{cases} \quad (2.1)$$

For  $\mu > 0$ , we set

$$X_\mu = \{(f, g) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) : N(f, g) = \|f\|_2^2 + \|g\|_2^2 = \mu\}$$

and consider the minimization problem  $m(\mu) = \inf\{\tau(f, g) : (f, g) \in X_\mu\}$ , where

$$\tau(f, g) = -a \int f'^2 - c \int g'^2 + \int f^2 g + 2 \int f g.$$

*Proposition 2.1.* For all  $\mu > 0$ ,  $m(\mu) > -\infty$ .

More precisely,  $m(\mu) \geq -\frac{C}{\sqrt[3]{|a|}} \mu^{\frac{10}{3}} - 2\mu$ , where  $C$  is a positive constant.

**Proof** Let  $(f, g) \in X_\mu$ . One only has to notice that by the Cauchy-Schwarz and Gagliardo-Nirenberg inequalities,

$$\begin{aligned} \left| \int f g \right| &\leq \|f\|_2 \|g\|_2 \leq \mu \quad \text{and} \\ \left| \int f^2 g \right| &\leq \|f\|_4^2 \|g\|_2 \leq \|f'\|_2^{\frac{1}{2}} \|f\|_2^{\frac{3}{2}} \|g\|_2 \leq \mu^{\frac{5}{2}} \|f'\|_2^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\tau(f, g) \geq -a \|f'\|_2^2 - \mu^{\frac{5}{2}} \|f'\|_2^{\frac{1}{2}} - 2\mu = P(\|f'\|_2),$$

which is enough to conclude since  $\frac{1}{\sqrt[3]{|a|}} \left(16^{-\frac{1}{3}} - 4^{-\frac{1}{3}}\right) \mu^{\frac{10}{3}} - 2\mu$  is the minimum of  $P(x) = -ax^4 - \mu^{\frac{5}{2}}x - 2\mu$ .  $\blacksquare$

*Proposition 2.2.* For all  $\mu > 0$ ,  $m(\mu) < -\frac{C}{\sqrt[3]{|a|+|c|}} \mu^{\frac{5}{3}}$ , where  $C$  is a positive constant.

**Proof** We fix a non-negative function  $h \in H^1(\mathbb{R})$  such that  $\|h\|_2 = 1$  and we put  $h_\lambda(x) = \lambda h(\lambda^2 x)$ , where  $\lambda$  will be chosen later. Then, for all  $\mu > 0$ ,  $(f, g) = \frac{1}{\sqrt{2}} \left( \mu^{\frac{2}{3}} h_\lambda(\mu^{\frac{1}{3}} x), -\mu^{\frac{2}{3}} h_\lambda(\mu^{\frac{1}{3}} x) \right) \in X_\mu$  and

$$\begin{aligned} m(\mu) \leq \tau(f, g) &= \frac{|a|+|c|}{2} \mu^{\frac{5}{3}} \int (h'_\lambda)^2 - \frac{1}{2\sqrt{2}} \mu^{\frac{5}{3}} \int h_\lambda^3 - \mu \int h_\lambda^2 \\ &\leq \mu^{\frac{5}{3}} \left( \frac{|a|+|c|}{2} \|h'_\lambda\|_2^2 - \frac{1}{2\sqrt{2}} \|h_\lambda\|_3^3 \right) \\ &\leq \mu^{\frac{5}{3}} \lambda \left( \lambda^3 (|a|+|c|) \|h'\|_2^2 - \frac{1}{2\sqrt{2}} \|h\|_3^3 \right). \end{aligned}$$

We conclude the proof by choosing  $\lambda = \frac{\epsilon}{\sqrt[3]{|a|+|c|}}$ , with  $\epsilon$  such that  $-C = \lambda^3 (|a|+|c|) \|h'\|_2^2 - \frac{1}{2\sqrt{2}} \|h\|_3^3 < 0$ .  $\blacksquare$

### 3. Existence of Minimizers

Let  $\mu > 0$  and  $(f_n, g_n)$  a minimizing sequence for  $m(\mu)$ . By denoting  $f^*$  the Schwarz symmetrization of  $|f|$ , it is well known that

$$\|f^{*'}\|_2 \leq \|f'\|_2, \quad \|f^*\|_2 = \|f\|_2, \quad \int f g \leq \int f^* g^* \quad \text{and} \quad \int f^2 g \leq \int f^{*2} g^*.$$

Hence,  $\tau(f^*, -g^*) \leq \tau(f, g)$  and  $(f, g) \in X_\mu$  implies that  $(f^*, -g^*) \in X_\mu$ . Therefore we can choose a minimizing sequence  $(f_n, g_n)$  with  $f_n \geq 0$ ,  $g_n \leq 0$  and  $f_n, -g_n$  radially decreasing.

We will now apply the concentration-compactness method ([6],[7]) to prove the compacty, up to translations, of the sequence  $(f_n, g_n)$  in  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  strong.

Following this method, we set the concentration function  $\rho_n = f_n^2 + g_n^2$  and put  $Q_n(t) = \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} \rho_n$ . Also, we set  $Q(t) = \lim_{n \rightarrow +\infty} Q_n(t)$  and  $\Omega = \lim_{t \rightarrow +\infty} Q(t)$ .

We start by ruling out vanishing:

*Proposition 3.1.* There exists  $\mu_0 = \mu_0(a, c)$  such that for  $\mu \geq \mu_0$ ,  $\Omega > 0$ .

**Proof** Assume that  $\Omega = 0$ . Since  $Q(t)$  is non-negative and non-increasing, for all  $t$ ,  $Q(t) = 0$ . Hence,

$$\lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} f_n^2 = \lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} g_n^2 = 0.$$

From the Proof of 2.1 one can infer that  $(f_n)$  (and  $g_n$ ) is bounded in  $H^1(\mathbb{R})$ . Arguing as in [6] (Lemma I.1),  $\|f_n\|_4 \rightarrow 0$ , hence  $\int f_n^2 g_n \rightarrow 0$  by Cauchy-Schwarz. Furthermore,

$$\begin{aligned} m(\mu) &= \lim_{n \rightarrow +\infty} \left( -a \int f_n'^2 - c \int g_n'^2 + \int f_n^2 g_n + 2 \int f_n g_n \right) \\ &\geq 2 \lim_{n \rightarrow +\infty} \int f_n g_n \geq -2\mu, \end{aligned}$$

which contradicts Proposition 2.2 for  $\mu \geq \mu_0 = \frac{2^{\frac{3}{2}}}{C_1^{\frac{3}{2}}} \sqrt{|a| + |c|}$ . ■

Next, we rule out dichotomy, that is  $0 < \Omega < \lim_{n \rightarrow +\infty} \int \rho_n$ . It is sufficient to prove the following lemma:

*Lemma 3.2.* For all  $\mu \geq \mu_0$  and for all  $\theta > 1$ ,  $m(\theta\mu) < \theta m(\mu)$ .

**Proof** We have  $\tau(\theta^{\frac{1}{2}} f_n, \theta^{\frac{1}{2}} g_n) = \theta \tau(f_n, g_n) - (\theta^{\frac{3}{2}} - \theta) \int |f_n|^2 |g_n|$ .

Also, there exists  $\delta > 0$  such that for all  $n$  large enough,  $\int |f_n|^2 |g_n| \geq \delta$ . Otherwise, up to a subsequence,  $\int f_n^2 g_n \rightarrow 0$ , which is absurd for  $\mu \geq \mu_0$ , as seen in the previous proof of Proposition 3.1. Finally,

$$m(\theta\mu) \leq \tau(\theta^{\frac{1}{2}} f_n, \theta^{\frac{1}{2}} g_n) \leq \theta \tau(f_n, g_n) - \delta(\theta^{\frac{3}{2}} - \theta),$$

which yields the result:

$$m(\theta\mu) \leq \lim_{n \rightarrow +\infty} \theta \tau(f_n, g_n) - \delta(\theta^{\frac{3}{2}} - \theta) = \theta m(\mu) - \delta(\theta^{\frac{3}{2}} - \theta) < \theta m(\mu). \quad \blacksquare$$

It is standard, from Lemma 3.2, to prove the strict subadditivity of  $m$ , that is

$$\forall \mu \geq \Omega, \quad m(\mu) < m(\Omega) + m(\mu - \Omega),$$

(see for instance Lemma 2.3 in [8]) which is well-known to rule out dichotomy.

Hence, by Lions' Theorem, we are in the compactness situation. There exists a sequence  $(y_n)$  such that, up to a subsequence,  $(f_n(\cdot - y_n), g_n(\cdot - y_n))$  converges strongly in  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  to some  $(\tilde{f}, \tilde{g}) \in X_\mu$ .

Since  $(f_n, g_n)$  is bounded in  $H^1(\mathbb{R})$ , using the compact embedding

$$H_{rad}^1(\mathbb{R}) \hookrightarrow L^4(\mathbb{R}),$$

up to a subsequence,  $f_n$  (respectively  $-g_n$ ) converges strongly in  $L^4$  to some radial non-negative function  $f$  (respectively  $-g$ ).

Furthermore,  $(f_n, g_n) \rightharpoonup (f, g)$  in  $H^1(\mathbb{R}) \times H^1(\mathbb{R})$  weak.

Since

$$\|f_n\|_2^2 + \|g_n\|_2^2 = \|f_n(\cdot - y_n)\|_2^2 + \|g_n(\cdot - y_n)\|_2^2 \xrightarrow{n \rightarrow +\infty} \|\tilde{f}\|_2^2 + \|\tilde{g}\|_2^2 = \mu,$$

we have in fact that  $(f_n, g_n) \rightarrow (f, g)$  in  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  strong and, in particular,  $(f, g) \in X_\mu$ . Hence,

$$\begin{aligned} \int f_n g_n &\rightarrow \int f^2 g, \quad \int f_n g_n \rightarrow \int f g \text{ and} \\ \lim_{n \rightarrow +\infty} |a| \int f_n'^2 + |c| \int g_n'^2 &\leq |a| \int f'^2 + |c| \int g'^2. \end{aligned}$$

From these inequalities, we deduce that  $\tau(f, g) \leq \lim_{n \rightarrow +\infty} \tau(f_n, g_n) = m(\mu)$ .

Since  $(f, g) \in X_\mu$ , we have in fact  $\tau(f, g) = m(\mu)$  and  $(f, g)$  is a minimizer for  $m(\mu)$ .

#### 4. End of the Proof of Theorem 1.1

There exists a Lagrange multiplier  $\lambda \in \mathbb{R}$  such that  $\nabla \tau(f, g) = \lambda \nabla N(f, g)$ , that is

$$\begin{cases} a f'' + f g + g &= \lambda f \\ c g'' + \frac{1}{2} f^2 + f &= \lambda g. \end{cases} \quad (4.1)$$

Multiplying these equations by  $f$  and  $g$  respectively and integrating by parts leads to

$$\begin{cases} -a \int (f')^2 + \int f^2 g + \int f g &= \lambda \int f^2 \\ -c \int (g')^2 + \frac{1}{2} \int f^2 g + \int f g &= \lambda \int g^2, \end{cases}$$

and, adding the equalities,

$$-\lambda \mu = -\tau(f, g) - \frac{1}{2} \int f^2 g \geq -\tau(f, g) = -m(\mu). \quad (4.2)$$

Note that, in particular,  $\lambda < 0$ . Setting

$$\phi(x) = -\frac{1}{\lambda} f \left( \frac{x}{\sqrt{-\lambda}} \right) \quad \text{and} \quad \psi(x) = -\frac{1}{\lambda} g \left( \frac{x}{\sqrt{-\lambda}} \right),$$

we obtain

$$\begin{cases} a\phi'' + \phi + \phi\psi - \frac{1}{\lambda}\psi &= 0 \\ c\psi'' + \psi + \frac{1}{2}\phi^2 - \frac{1}{\lambda}\phi &= 0, \end{cases} \quad (4.3)$$

that is,  $(\phi, \psi)$  is a solution to (2.1) with speed  $\omega = \frac{1}{\lambda}$ .

To obtain the  $L^2$  size of this solution, a simple computation shows that

$$\|\phi\|_2^2 + \|\psi\|_2^2 = \frac{\mu}{|\lambda|^{\frac{3}{2}}}.$$

Also, from (4.2) and Proposition 2.2,  $|\lambda|\mu \geq -m(\mu) \geq \frac{C}{\sqrt[3]{|a|+|c|}}\mu^{\frac{5}{3}}$ , from where we conclude that

$$\|\phi\|_2^2 + \|\psi\|_2^2 \leq C\sqrt{|a|+|c|},$$

where  $C$  is yet another positive constant independent of  $a$ ,  $c$  and  $\mu$ . Also, note that

$$|\omega| = \frac{1}{|\lambda|} \leq \frac{1}{C_1} \sqrt[3]{|a|+|c|} \mu^{-\frac{2}{3}}.$$

The regularity of  $(\phi, \psi)$  can be obtained by a standard bootstrapping argument (see for instance [4], Proposition 3.2).

In order to prove the exponential decay of  $\phi$  and  $\psi$ , following the ideas of Theorem 8.1.1 in [2], we consider, for  $\epsilon, \eta > 0$ ,  $h(x) = e^{\frac{\epsilon|x|}{1+\eta|x|}} \in L^\infty(\mathbb{R})$ . Multiplying equations in (2.1) by  $h\phi$  and  $h\psi$  respectively and integrating, we get

$$a \int h\phi\phi'' + c \int h\psi\psi'' + \int h(\phi^2 + \psi^2) + \frac{3}{2} \int h\phi^2\psi - 2\omega \int h\phi\psi = 0$$

Integrating by parts and using the fact that  $h' \leq \epsilon h$ , we obtain

$$\int h(\phi^2 + \psi^2 - 2\omega\phi\psi) \leq a \int h\phi'^2 + c \int h\psi'^2 + \epsilon \int h(|\phi\phi'| + |\psi\psi'|) + \frac{3}{2} \int h\phi^2|\psi|,$$

and

$$\begin{aligned} \int h \left( \left(1 - \frac{\epsilon}{2}\right) \phi^2 + \left(1 - \frac{\epsilon}{2}\right) \psi^2 - 2\omega\phi\psi \right) \leq \\ \left(a + \frac{\epsilon}{2}\right) \int h\phi'^2 + \left(c + \frac{\epsilon}{2}\right) \int h\psi'^2 + \frac{3}{2} \int h\phi^2|\psi| \leq \frac{3}{2} \int h\phi^2|\psi| \end{aligned}$$

for  $\epsilon$  small enough. Since  $\psi \in H^1(\mathbb{R})$ ,  $\lim_{|x| \rightarrow +\infty} \psi(x) = 0$ . For  $\epsilon' > 0$  to be chosen later, we set  $r > 0$  such that  $|\psi(x)| \leq \epsilon'$  for  $|x| > r$ . We then get

$$\int h \left( \left(1 - \frac{\epsilon}{2} - \frac{3\epsilon_1}{2}\right) \phi^2 + \left(1 - \frac{\epsilon}{2}\right) \psi^2 - 2\omega\phi\psi \right) \leq \frac{3}{2} \int_{|x| \leq r} h\phi^2|\psi|$$

and

$$\int h(C_1\phi^2 + C_2\psi^2) \leq \frac{3}{2} \int_{|x| \leq r} h\phi^2|\psi|$$

where  $C_1 = 1 - \frac{\epsilon}{2} - \frac{3\epsilon_1}{2} - |\omega| > 0$  and  $C_2 = 1 - \frac{\epsilon}{2} - |\omega| > 0$  for  $\epsilon, \epsilon'$  small enough (and for  $|\omega|$  small).

Finally, taking  $\eta \rightarrow 0$ , by Fatou's Lemma and Lebesgue's Theorem, we obtain

$$\int e^{\epsilon|x|}\phi^2 < +\infty \text{ and } \int e^{\epsilon|x|}\phi^2 < +\infty.$$

In view of Theorem 8.1.7 of [2], this is enough to conclude that

$$e^{\alpha|x|}\phi, e^{\alpha|x|}\psi \in L^\infty(\mathbb{R})$$

for some  $0 < \alpha \leq \epsilon$ .

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